The stability of ferromagnetism in the Hubbard model on two-dimensional line graphs

2009 J. Phys. A: Math. Theor. 42 265002

(http://iopscience.iop.org/1751-8121/42/26/265002)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 265002 (10pp) doi:10.1088/1751-8113/42/26/265002

The stability of ferromagnetism in the Hubbard model on two-dimensional line graphs

Long Lu

Institute für Theoretische Physik, Universität Heidelberg, Philosophenweg 19, D-69120, Heidelberg, Germany

Received 4 March 2009, in final form 11 May 2009
Published 9 June 2009
Online at stacks.iop.org/JPhysA/42/265002

Abstract

The Hubbard model on line graphs has a flat band and highly degenerate ferromagnetic ground states. Here, we study the Hubbard model on a line graph of a planar bipartite graph by adding a special contribution to the kinetic energy which lifts the degeneracy of the lowest single particle state to a general Hamiltonian. We prove that, at half-filling of the lowest band and for sufficiently strong repulsion U, the ground states of this Hamiltonian remain saturated ferromagnetic for a class of line graphs of planar bipartite graphs.

PACS numbers: 05.50.+q, 71.10.Fd

1. Introduction

The origin of ferromagnetism has been an old and not yet solved problem in physics for quite a long time. A modern version of the mechanism for itinerant electron ferromagnetism is formulated by Kanamori [1] and Hubbard [2]. The model is usually called the Hubbard model, which established the generation of ferromagnetism in some simplified situations. Unfortunately, due to the complexity of the Hubbard model, rigorous results are singular, which have infinite repulsive Coulomb interaction [3] or in which there exists a dispersionless band generating magnetization [4] and the flat-band ferromagnetism formulated by Mielke [5] and Tasaki [6]. It was often argued that the flatness of the band in the flat-band model is artificial and that flat bands do not occur in nature; therefore, the next important step toward the understanding of ferromagnetism is to show the stability of the flat-band ferromagnetism against the perturbation breaking of the flatness of the band. Tasaki himself has successfully proved the stability of the flat-band ferromagnetism in rather general situations for his models [7, 8]. The main problem in Tasaki's model is that there is a large band gap between the lowest nearly flat band and the higher bands. Since the lowest band is half-filled and the Coulomb interaction U is large, one might have a Mott insulator in that case. As for Mielke's models, only the stability of the flat-band ferromagnetism on the kagomé lattice was established by Tanaka and Ueda [9].

In this paper, we study the stability of ferromagnetism in the Hubbard model on the line graphs of planar bipartite graphs in two dimensions. We add a special perturbation to the hopping parts of the Hamiltonian to get a dispersion band and show the stability of the flat-band ferromagnetism on them. The kagomé lattice discussed in [9] is a special case in this larger class of lattices. This paper is organized as follows. In section 2, we give some definitions and state our main theorem. The proof of the theorem and some discussions are given in section 3.

2. Definitions and result

2.1. Definitions

Let us first introduce some definitions and properties concerning the line graphs of planar bipartite graphs. We use the notation in [11].

A graph is a collection of vertices and the edges between them and will be denoted by $G = (V, E)$, where V is the set of vertices and E is the set of edges. $|V|$ is the number of vertices and $|E|$ is the number of edges. A walk of length $n-1$ is a sequence $c = \{x_1, e_1, x_2, e_2, \ldots, e_{n-1}, x_n\}$, where e_i is an edge joining x_i and x_{i+1}. A path is a self-avoiding walk and a cycle is a self-avoiding closed walk.

Let G be a bipartite planar 2-connected graph. G is said to be bipartite if the vertices of G can be divided into two disjoint sets V_1 and V_2 such that each edge joins a vertex from V_1 to V_2. In a bipartite graph, the length of each cycle is even. G is said to be 2-connected if no edge exists such that the graph decays into two unconnected subgraphs if this edge is deleted from G. In a 2-connected graph, each edge is contained in a cycle. A planar graph has a representation in the plane as a set of points and lines so that there are no lines intersecting each other.

The edges and vertices of a planar graph divide the plane into a set of connected components, called faces. Each plane graph has exactly one unbounded face. Let $F(G)$ be the set of bounded faces of G. One has $|F(G)| = |E(G)| - |V(G)| + 1$ due to the Euler theorem. The boundary of each bounded face f is called a facial cycle.

The line graph $L(G)$ has the edges of G as vertices and two vertices are connected by an edge, if the corresponding edges in G have a vertex in common. For instance, one may take the hexagonal lattice (or a finite part of it, e.g., with periodic boundary conditions). The line graph of it is the kagomé lattice. It is a planar graph. Another example is the line graph of the square lattice with periodic boundary conditions. This graph may be represented by a regular lattice of corner-sharing tetrahedra. Note that this graph is not planar.

We define the adjacency matrix of the graph G to be the matrix $A = (a_{xy})_{x,y\in V(E)}$ where $a_{xy} = 1$ if the two vertices are adjacent and $a_{xy} = 0$ otherwise. Further, the vertex-edge-incidence matrix is defined by $B = (b_{xe})_{x\in V(G),e\in E(G)}$. $b_{xe} = 1$ if vertex x belongs to edge e and $b_{xe} = 0$ otherwise. The face-edge-incidence matrix is defined by $\tilde{B} = (\tilde{b}_{fe})_{f\in F(G),e\in E(G)}$. $\tilde{b}_{fe} = 1$ if edge e belongs to the boundary of f and $\tilde{b}_{fe} = 0$ otherwise. The number of edges of the boundary of f is $n_f = \sum_{e\in E} \tilde{b}_{fe}$. Since the graph is bipartite, n_f is even. The vertex-face-incidence matrix is defined by $C = (c_{xf})_{x\in V(G),f\in F(G)}$. $c_{xf} = 1$ if the vertex x belongs to the face f and $c_{xf} = 0$ otherwise. One has $c_{xf} = \frac{1}{2}\sum_{e\in E(G)} \tilde{b}_{fe} b_{xe}$. We let $d_x = \sum_{f\in F(G)} c_{xf}$ and $D = \mathrm{diag}(d_x)$. d_x is the number of bounded faces touching the vertex x.

Since G is bipartite with vertex sets V_1 and V_2, each edge in G may be oriented from V_1 to V_2. Furthermore, the facial cycle of f will be oriented clockwise. We now define $S = (s_{fe})_{f\in F(G),e\in E(G)}$, where $s_{fe} = 1$ if f contains e, e and the facial cycle of f has the same

orientation. $s_{fe} = -1$ if f contains e, e and the facial cycle of f has the opposite orientation. $s_{fe} = 0$ otherwise.

The adjacency matrix A_L of a line graph $L(G)$ has the form

$$A_L = B^T(G)B(G) - 2I, \tag{1}$$

where B^T is the transposition of B and I denotes the unit matrix. The lowest eigenvalue of A_L is -2. Since $B^T(G)B(G)$ is a positive-semidefinite matrix, it follows from (1) that each eigenvalue a_L of the adjacency matrix A_L obeys $a_L \geqslant -2$. Each element of the kernel of $B(G)$ is an eigenstate to this eigenvalue. One has further in the graph theory $\sum_e b_{xe}s_{fe} = 0$. It means that the columns of S^T forms a basis of the kernel of B. Furthermore, the dimension of the kernel of B is $|F(G)|$. This number is thus the multiplicity of the eigenvalues -2 of A_L.

2.2. The Hamiltonian

Let us now define the Hubbard model on $L(G)$. The Hamiltonian of the Hubbard model on a graph G has the form

$$H = \sum_{e,e' \in E(G),\sigma} t_{ee'} c^\dagger_{e\sigma} c_{e'\sigma} + \sum_{e \in E(G)} U n_{e\uparrow} n_{e\downarrow}, \tag{2}$$

where

$$t_{ee'} = t \sum_{x \in V(G)} b_{xe} b_{xe'} - \sum_{f \in F(G)} \frac{s_f}{n_f} s_{fe} s_{fe'}. \tag{3}$$

Let us first make some notes.

- We assume that $t > 0, s_f > 0$ and $U > 0$. $c^\dagger_{e\sigma}$ and $c_{e\sigma}$ are the usual creation and the annihilation operators of an electron on the vertices of $L(G)$, respectively. They satisfy the fermion anticommutation relations. The number operator for the electrons on an edge e with the spin σ is defined as $n_{e\sigma} = c^\dagger_{e\sigma} c_{e\sigma}$. $N = \sum_{e\sigma} n_{e\sigma}$ is the number of electrons. We consider many-electron states with the total electron number fixed at $|F(G)|$. The present number corresponds to the half-filling of the lowest band.
- The first part of the Hamiltonian describes the hopping of electrons on $L(G)$ and the second part is the on-site repulsive Coulomb interaction. The first part of $t_{ee'}$ is the usual nearest-neighbor hopping on $L(G)$ plus an additional term $2t\delta_{ee'}$. The lowest eigenvalue of $\sum_x b_{xe} b_{xe'}$ is zero. If we set $s_f = 0$, we get the flat-band ferromagnetism.
- The second added term of $t_{ee'}$ changes the flat lowest band into a dispersive band and thus lifts the degeneracy of the lowest eigenvalue of $\sum_x b_{xe} b_{xe'}$, but it does not change the other eigenvalues. The added term is negative semi-definite; all eigenvalues which are 0 for $s_f = 0$ become non-positive. It can also be written as $H_1 = -s_f \sum_{f\sigma} d^\dagger_{f\sigma} d_{f\sigma}$, where $d^\dagger_{f\sigma} = n_f^{-1/2} \sum_e s_{fe} c^\dagger_{e\sigma}$. Since the corresponding single particle states are not orthogonal, $\tilde{s}_{ff'} = \left\{ d_{f\sigma}, d^\dagger_{f'\sigma} \right\} = (n_f n_{f'})^{-1/2} \sum_{e,e'} s_{fe} s_{f'e'}$ is not a diagonal matrix. Let $S_F = \mathrm{diag}(s_f)$ and $\tilde{S} = (\tilde{s}_{fg})_{f,g \in F(G)}$. Then the single particle spectrum of H_1 is given by the spectrum of the matrix $-\sqrt{S_F}\tilde{S}\sqrt{S_F}$.

2.3. The theorem

We investigate the line graphs of planar bipartite graphs. The lattices are characterized by n_f. Since the graph is bipartite, n_f is even. We assume that the coordination number $d = 3$ is fixed. This means that every vertex belongs to three edges; two of them are the boundary of a

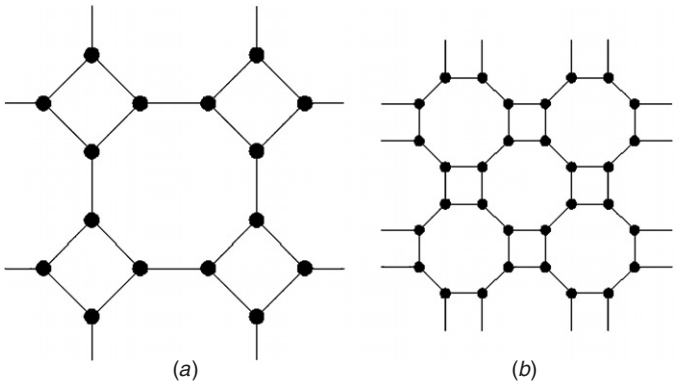

Figure 1. Examples for planar bipartite graphs with a fixed coordination number 3. (*a*) Quadrangles. (*b*) Octagons.

face f and we will call the third one the outer edge of f. The regular graph G consisting of quadrangles and octagons is a good example in our model (figure 1).

We can now state our main theorem.

Theorem. *Let G be a bipartite planar 2-connected graph and assume that, for each vertex $x \in V(G)$, $\sum_{e\in E(G)} b_{xe}$ equals 3 if x is included in a facial cycle of a bounded face of G and 1 otherwise. Furthermore, we assume $n_f \leqslant 10$ for each bounded face f of G. Consider the Hubbard model on a 2-connected planar line graph $L(G)$ with the electron number $N = |F(G)|$. Then, for sufficiently large $t > t_c$ and $U > U_c$, both independent of the lattice sites and finite in the thermodynamic limit, the ground state of the Hamiltonian (2) is fully polarized $S = \frac{N}{2}$. Furthermore, the ground state is unique up to the $2S + 1$-fold degeneracy due to the $SU(2)$ symmetry.*

Remarks.

- We note the following essential difference between our model and Tasaki's models [8, 10]: there are no band gaps in our model while there are finite or infinite band gaps in Tasaki's models, which lead to the Mott insulator or metallic ferromagnetism, respectively. In [10] the conductivity is due to the electrons in the higher band, which is partially filled, are movable and are coupled ferromagnetically to the electrons in the lower band. In our model, the fact that the single particle eigenstates are an extended state and that the Fermi surface does not lie in a band gap imply this model is also a candidate for a ferromagnetic metal.
- It can be proved that the model with a nonflat band exhibits Pauli paramagnetism for $U = 0$ and can be strongly believed that, for sufficiently small U, the ground states of the model are spin singlet. Therefore, one must have sufficiently large U to get ferromagnetism.
- The proof of the theorem makes use of the fact that the columns of S^T form a basis of the kernel of B. One might expect that it should be possible to extend this result to $N < |F(G)|$. But a simple generalization of the theorem is not possible. The reason is that we show $H \geqslant -\sum_f s_f$ for sufficiently large U and t together with the fact that the ferromagnetic state $\prod_f d^\dagger_{f\uparrow}|0\rangle$ has the energy $-\sum_f s_f$. Since the state $d^\dagger_{f\sigma}|0\rangle$ is local, but the single particle ground states of H are not local, therefore it is not possible to express the ferromagnetic eigenstates of H with lowest energy by local operators for $N < |F(G)|$.

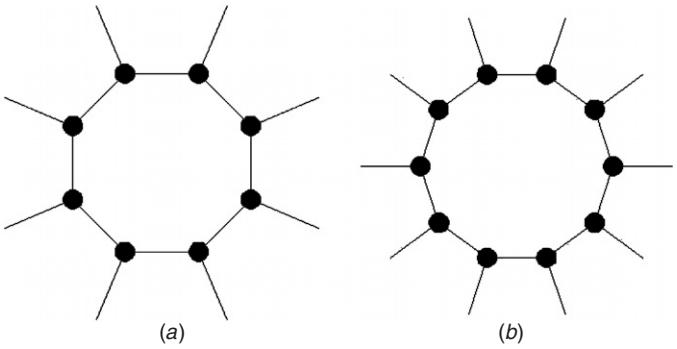

Figure 2. Subgraph g_f. (*a*) $n_f = 8$. (*b*) $n_f = 10$.

- It is not possible to easily generalize the result to high dimensional lattices. The reason is that one can construct a basis using small cycles on G, similar to the boundaries of the faces of the planar graph, but the number of small cycles is much larger than the number of single particle ground states. As a consequence, the bounds we use in the proof do not hold.

3. Proof of the theorem

Since the case $n_f = 4, 6$ can be dealt with through the method of Tanaka and Ueda, the remaining task to prove the above theorem is to investigate the cases $n_f = 8, 10$.

To prove the theorem, we first establish ferromagnetism in a local model described by H_f and then show that this local ferromagnetism can be connected. The Hamiltonian H can then be written as a sum of local Hamiltonians $H = \sum_{f\in F(G)} H_f$, where

$$H_f = H_{f,1} + H_{f,2} + H_{f,3} \tag{4}$$

with

$$H_{f,1} = -\frac{s_f}{n_f} \sum_{e,e'\in E(G),\sigma} s_{fe} c^\dagger_{e\sigma} s_{fe'} c_{e'\sigma} \tag{5}$$

$$H_{f,2} = t \sum_{x\in V(G)} \frac{c_{xf}}{d_x} \sum_{e,e'\in E(G),\sigma} b_{xe} c^\dagger_{e\sigma} b_{xe'} c_{e'\sigma} \tag{6}$$

$$H_{f,3} = \sum_{x\in V(G)} \frac{c_{xf}}{2d_x} \sum_{e\in E(G)} b_{xe} U n_{e\uparrow} n_{e\downarrow}. \tag{7}$$

Let g_f be a subgraph (figure 2) consisting of a facial cycle and outer edges of a bounded face of f. We consider the eigenvalue problem corresponding to the Hamiltonian H_f restricted on the line graph $L(g_f)$ of g_f.

In the limit $t = \infty$ and $U = \infty$, $H_{f,2} \geqslant 0$ (6) means that the single Hilbert space is restricted to single particle states with $H_{f,2}\Phi = 0$. If we define $2n_f \times 2n_f$ matrices B_f and C_f as

$$B_f = (b_{xe})_{x\in V(g_f), e\in E(g_f)} \tag{8}$$

and

$$C_f = \mathrm{diag}(c_{xf})_{x\in V(g_f)}, \tag{9}$$

respectively, then $H_{f,2}$ restricted on $L(g_f)$ is equivalent to the matrix

$$H_{f,2} = tB_f^T C_f D^{-1} C_f B_f. \tag{10}$$

Then, the kernel of $H_{f,2}$ is the kernel of $C_f B_f$. It is easy to check that the rank of $C_f B_f$ is n_f and thus the dimension of the kernel of $C_f B_f (H_{f,2})$ is also n_f.

We introduce our basis states. We number the outer edges of f as $e_1, \ldots, e_{n_f}$. We let the path p_i start at e_i and end at e_{i+1} (we denote $n_f + 1 = 1$). The path is oriented. Now we let $\phi_i(e) = 1$ if e belongs to p_i and has the same orientation as p_i, $\phi_i(e) = -1$ if e belongs to p_i and has the opposite orientation, and $\phi_i(e) = 0$ if e does not belong to p_i. Obviously, the n_f states we constructed are linearly independent and nonorthogonal. Furthermore, they form a basis of the kernel of $H_{f,2}$ (6).

We define fermion operators by using the above-defined states $a_{i\sigma} = \sum_{e\in E(G)} \phi_i(e) c_{e\sigma}$. Let Φ be a multi-particle state which is an eigenstate of $H_{f,2}$ with eigenvalue 0 in the limit $U, t = \infty$. It can be expanded as

$$\Phi = \sum_{I_\uparrow, I_\downarrow \subset I} g(I_\uparrow; I_\downarrow) \Phi(I_\uparrow; I_\downarrow) \tag{11}$$

with complex coefficients $g(I_\uparrow; I_\downarrow)$, where

$$\Phi(I_\uparrow; I_\downarrow) = \prod_{p\in I_\uparrow} a^\dagger_{p\uparrow} \prod_{p'\in I_\downarrow} a^\dagger_{p'\downarrow} \Phi_0. \tag{12}$$

Φ_0 is a state with no electrons. We denote $I = \{1, 2, \ldots, n_f\}$. Since the on-site interaction $U = \infty$, any ground state Φ must further satisfy

$$\sum_{I_\uparrow, I_\downarrow \subset I} g(I_\uparrow; I_\downarrow) c_{e\uparrow} c_{e\downarrow} \Phi(I_\uparrow; I_\downarrow) = 0. \tag{13}$$

The advantage of using this kind of basis is that the finite energy condition (13) can be considered explicitly. This means that we can select the states which satisfy (13) exactly and evaluate numerically the minimum energy of the states.

We first solve the 2-electron problem. Since the local spin operators

$$S'_z = \frac{1}{2} \sum_{e\in E(g_f)} (n_{e\uparrow} - n_{e\downarrow}), \qquad S'_+ = \sum_{e\in E(g_f)} c^\dagger_{e\uparrow} c_{e\downarrow}, \qquad S'_- = \sum_{e\in E(g_f)} c^\dagger_{e\downarrow} c_{e\uparrow} \tag{14}$$

and

$$S'^2 = \tfrac{1}{2}(S'_+ S'_- + S'_- S'_+) + S'^2_z \tag{15}$$

commute with $H_{f,2}$, S'^2 and S'_z are conserved. We work in the $S'_z = 0$ subspace since all competitors have a representative there. This is to say that each eigenstate with a given S'_z can be rotated in a spin space to a state with $S'_z = 0$ without changing its energy. Then a 2-electron state with $S'_z = 0$ can be written as

$$\Phi_2 = \sum_{i,j\in I} g_{ij} a^\dagger_{i\uparrow} a^\dagger_{j\downarrow} \Phi_0 \tag{16}$$

with g_{ij} as coefficients.

The finite energy condition (13) is given by

$$g_{ii} = 0, \qquad g_{i,i\pm1} + g_{i\pm1,i} = 0, \qquad \forall\, i \in I. \tag{17}$$

The energy expectation value of Φ_2 is

$$E_2 = \frac{\langle \Phi_2 | H_f | \Phi_2 \rangle}{\langle \Phi_2 | \Phi_2 \rangle} = -\frac{s_f}{n_f} \cdot \frac{2 \sum_{i,j,k,l=1}^{n_f} g^\star_{ij} g_{kl} (-1)^{i+k} A_{jl}}{\sum_{i,j,k,l=1}^{n_f} g^\star_{ij} g_{kl} A_{ik} A_{jl}} \tag{18}$$

with $A_{ij} = 3\delta_{ij} + \delta_{i,j+1} + \delta_{i,j-1}$ (we denote $n_f + 1 = 1$).

We numerically evaluate the minimum energy (18) within the constraints (17) and the normalization condition

$$\sum_{i,j,k,l=1}^{n_f} g^{\star}_{ij} g_{kl} A_{ik} A_{jl} = 1. \tag{19}$$

The result shows that the minimum energy $E_{\min} = -s_f$ for $n_f \leqslant 10$.

It is noted that $E_{\min} < -s_f$ for $n_f \geqslant 12$. This shows the instability of ferromagnetism on the local graph with $n_f \geqslant 12$. However, the instability of ferromagnetism on the local graph does not imply that on the entire graph in general. This means that the prove strategy by using the local Hamiltonian H_f is not suitable to estimate the stability of the flat-band ferromagnetism on line graphs with $n_f \geqslant 12$. A new prove method is needed to study the stability of ferromagnetism on them.

Now assume that $\Phi_n = \sum_{I_\uparrow, I_\downarrow \subset I} g(I_\uparrow; I_\downarrow)\Phi(I_\uparrow; I_\downarrow)$ is an n-electron state with $|I_\uparrow|+|I_\downarrow| = n$. The energy expectation value is expressed as

$$E_n = \frac{\langle \Phi_n | H_f | \Phi_n \rangle}{\langle \Phi_n | \Phi_n \rangle} \tag{20}$$

with

$$\langle \Phi_n | H_f | \Phi_n \rangle = -\frac{s_f}{n_f} \sum_{I_\uparrow, I_\downarrow, I'_\uparrow, I'_\downarrow \subset I} g(I_\uparrow; I_\downarrow) g(I'_\uparrow; I'_\downarrow) \Bigg(\sum_{x \in I_\uparrow, y \in I'_\uparrow} \left\langle \Phi\left(I'^{y}_{\uparrow}; I'_\downarrow\right) \middle| \Phi\left(I^{x}_{\uparrow}; I_\downarrow\right) \right\rangle + \sum_{x \in I_\downarrow, y \in I'_\downarrow} \left\langle \Phi\left(I'_\uparrow; I'^{y}_{\downarrow}\right) \middle| \Phi\left(I_\uparrow; I^{x}_{\downarrow}\right) \right\rangle \Bigg) \tag{21}$$

and

$$\langle \Phi_n | \Phi_n \rangle = \sum_{I_\uparrow, I_\downarrow, I'_\uparrow, I'_\downarrow \subset I} g(I_\uparrow; I_\downarrow) g(I'_\uparrow; I'_\downarrow) \langle \Phi(I'_\uparrow; I'_\downarrow) | \Phi(I_\uparrow; I_\downarrow) \rangle. \tag{22}$$

Furthermore, we find the finite energy conditions (13) for Φ_n:

$$g\left(I^i_\uparrow; I^i_\downarrow\right) = 0, \qquad \forall\, i \in I. \tag{23}$$

$$g\left(I^i_\uparrow; I^{i'}_\downarrow\right) + g\left(I^{i'}_\uparrow; I^i_\downarrow\right) = 0, \qquad \text{if} \quad a_{ii'} = 1, \tag{24}$$

where $a_{ii'} = 1$ if $i = i' \pm 1$ and 0 otherwise. Here, we abbreviate $I_\sigma \cup \{p\}$ as I^p_σ for $p \in I$. Together with the normalization condition $\langle \Phi_n | \Phi_n \rangle = 1$, the minimum energy for all possible number of electrons and corresponding spin configurations is evaluated numerically. The result shows that $E_{\min} = -s_f$ for all Φ_n with $n_f = 8, 10$.

Thus, we can conclude that $E_{\min} = -s_f$ for any Φ in the case $n_f = 8, 10$. One finds that such Φ with $E_{\min} = -s_f$ indeed exists by testing

$$\Phi = d^\dagger_{f\uparrow}\Phi_+ + d^\dagger_{f\downarrow}\Phi_- \tag{25}$$

with $d^\dagger_{f\sigma} = \frac{1}{\sqrt{n_f}} \sum_i (-1)^i a^\dagger_{i\sigma}$, and $\Phi_\pm$ are build by linear combinations of products of a suitable number of $a^\dagger_{i\sigma}$. Φ satisfies further the finite energy condition (13). We point out that a state which contains terms with doubly occupied $d^\dagger_{f,\sigma}$ is not a lowest energy state in the limit $t, U = \infty$. This can be seen as follows.

Let $\Psi_{n,s}$ be a general n-electron state with $S'_z = s$:

$$\Psi_{n,s} = \sum_{I_\uparrow, I_\downarrow \subset I, |I_\uparrow|-|I_\downarrow| = s, |I_\uparrow|+|I_\downarrow| = n} g(I_\uparrow; I_\downarrow) \prod_{p \in I_\uparrow} a^\dagger_{p\uparrow} \prod_{p' \in I_\downarrow} a^\dagger_{p'\downarrow} \Psi_0, \tag{26}$$

Table 1. The ground-state energy in different S' sectors (unit s_f).

Electron number	$S = S_z$						
N_e	0	1/2	1	3/2	2	5/2	3
3	–	−0.81	–	−1	–	–	–
4	−0.65	–	−0.81	–	−1	–	–
5	–	−0.67	–	−0.81	–	−1	–
6	−0.54	–	−0.65	–	−0.81	–	−1

where $g(I_\uparrow; I_\downarrow)$ is a coefficient and Ψ_0 is a state with no electrons. We use the S'_+ operator to project $\Psi_{n,s}$ into the sector $S' = S'_z$:

$$S'_+ \Psi_{n,s} = \sum_{e \in E(G)} c^\dagger_{e\uparrow} c_{e\downarrow} \Psi_{n,s} = 0. \tag{27}$$

We then obtain a necessary condition for $g(I_\uparrow; I_\downarrow)$ and together with the finite energy (13) and the normalization condition, suitable states $\Psi_{n,s}$ can be selected and the minimum energy will be evaluated numerically. The results for the cases $n_f = 8$ are listed in table 1. It shows that a state with large S' has lower energy. It also implies that a state which contains terms with doubly occupied $d^\dagger_{f,\sigma}$ with $S' = 0$ has higher energy; therefore, it could not be a lowest energy state. Thus, any lowest energy state in the limit $t, U = \infty$ can be written in form (25).

Furthermore, there exist energy gaps between the ground states of different S' sectors. This demonstrates the robustness of the ferromagnetism under the spin-flip process and verifies the theorem formulated by Mielke [13] that local stability implies the global stability.

The continuity of energy implies that such Φ (25) is also the lowest energy state of H_f for sufficiently large but finite U and t as well. On the other hand, the critical values of U and t do not change when one tends to the thermodynamic limit and are independent of the size of G.

The theorem is a consequence of the above result. We assume that the value of U and t are large enough to hold the above statement. One has $H \geqslant -\sum_{f \in F(G)} s_f = -N s_f$. We now consider a fully polarized state

$$\Phi_f = \prod_{f \in F(G)} d^\dagger_{f\uparrow} \Phi_0. \tag{28}$$

The operator $\tilde{d}_{g\sigma} = \sum_f (\tilde{S}^{-1})_{gf} d_{f\sigma}$ is the dual operator to $d_{f\sigma}$. We have $\{\tilde{d}_{f\sigma}, d^\dagger_{g\sigma}\} = \delta_{fg}$ and hence we obtain

$$\begin{aligned} H_{f,1}\Phi_f &= -\sum_{f \in F(G)} s_f d^\dagger_{f\uparrow} d_{f\uparrow} \Phi_f \\ &= -\sum_{f,g \in F(G)} s_f \tilde{s}_{fg} d^\dagger_{f\uparrow} \tilde{d}_{g\uparrow} \Phi_f \\ &= -\sum_{f \in F(G)} s_f \tilde{s}_{ff} \Phi_f \\ &= -\sum_{f \in F(G)} s_f \Phi_f \\ &= -N s_f \Phi_f. \end{aligned} \tag{29}$$

This shows that for sufficiently large U and t, the fully polarized state Φ_f and its $SU(2)$ rotations are the ground states of H (2).

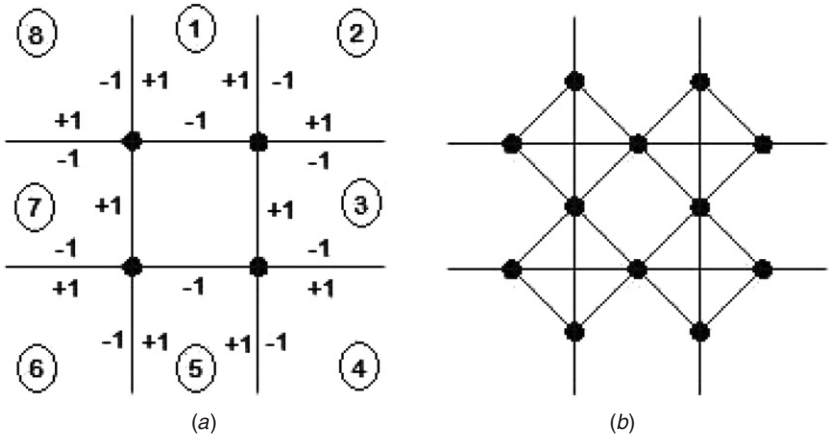

Figure 3. Quadratic lattice (*a*) and its line graph (*b*).

The proof of the uniqueness of Φ_f can be easily carried over from [12], since both operators $H_{f,1}$ and $H_{f,2}$ are commutative. This completes the proof of the theorem.

The stability of the ferromagnetic state on the line graph of a quadratic lattice (figure 3), which is a regular lattice of corner-sharing tetrahedra, can be also estimated by using a similar method discussed above, although its line graph is not planar.

We define our basis states on a local face f as follows. We number the outer edges as $e_1, \ldots, e_8$ and let p_i be the path that starts at e_i, ends at e_{i+1} and runs clockwise around f. We denote $I_1 = \{1, 4, 5, 8\}$ and $I_2 = \{2, 3, 6, 7\}$ as two subsets of $I = \{1, \ldots, 8\}$. We let $\phi_i(e) = 1, i \in I_1$, and $\phi_i(e) = -1, i \in I_2$, if e belongs to the path p_i and has the same orientation as p_i, and $\phi_i(e) = -1, i \in I_1$, and $\phi_i(e) = 1, i \in I_2$, if e belongs to the path p_i and has opposite orientation to p_i. $\phi_i(e) = 0, i \in I$, if e does not belong to p_i.

By using this kind of basis states together with the finite energy condition, the following evaluation shows that the minimum energy $E = -s_f$ for sufficiently strong repulsion U, which establishes the stability of the ferromagnetism on it. The proof of the uniqueness of the ground state can be carried from the proof of the bipartite graph without any changes.

4. Summary

We have been able to establish the stability of ferromagnetism by adding a special contribution to the kinetic energy which lifts the degeneracy of the lowest band on the line graphs of planar bipartite graphs with $n_f \leqslant 10$. The electron number we have chosen corresponds to the half-filling of the lowest nearly flat band. From the standard band-theoretic point of view, an electron system with such a filling becomes metallic. Due to the non-degeneracy of the band, no band gap and the extended single particle eigenstates, our model is a good candidate for metallic ferromagnetism. However, when we recall the Mermin–Wagner theorem, in dimensions one or two, ferromagnetism is inevitably destroyed by infinitesimally small thermal fluctuation. Therefore, in order to have ferromagnetism stable at finite temperatures, we must treat models in three dimensions. This is the next step toward understanding the origin of ferromagnetism.

Acknowledgments

I wish to thank Professor A Mielke for many useful and important suggestions and stimulating discussions which made this paper possible.

References

[1] Kanamori J 1963 *Prog. Theor. Phys.* **30** 275
[2] Hubbard J 1963 *Proc. R. Soc.* A **276** 238
[3] Nagaoka Y 1966 *Phys. Rev.* **147** 392
Thouless D J 1965 *Proc. Phys. Soc. London* **86** 893
[4] Lieb E H 1989 *Phys. Rev. Lett.* **62** 1201
[5] Mielke A 1991 *J. Phys. A: Math. Gen.* **24** L73
Mielke A 1991 *J. Phys. A: Math. Gen.* **24** 3311
Mielke A 1992 *J. Phys. A: Math. Gen.* **25** 4335
[6] Tasaki H 1992 *Phys. Rev. Lett.* **69** 1608
[7] Tasaki H 1996 *J. Stat. Phys.* **84** 535
[8] Tasaki H 1995 *Phys. Rev. Lett.* **75** 4678
[9] Tanaka A and Ueda H 2003 *Phys. Rev. Lett.* **90** 067204
[10] Tanaka A and Tasaki H 2007 *Phys. Rev. Lett.* **98** 116402
[11] Bollobás B 1979 *Graphy Theory* (Berlin: Springer)
[12] Mielke A 1993 *Phys. Lett.* A **174** 443–8
[13] Mielke A 1999 *Phys. Rev. Lett.* **82** 21